A Quant's Perspective on the 2008 Financial Crisis

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1 Introduction

These notes explore the 2008 financial crisis from a quantitative finance perspective, focusing on the mathematical models and financial instruments that contributed to the crisis. The analysis covers bond pricing, CDOs, correlation modeling, and the systemic risks that emerged.

2 Bond Pricing Fundamentals

2.1 Basic Bond Pricing Formula

For any bond with cash flows $C_1, C_2, \ldots, C_N$ at times $t_1, t_2, \ldots, t_N$ and interest rate curve $r(t)$:

$$\text{Bond Price} = \sum_{i=1}^{N} C_i e^{-r(t_i)\cdot t_i}$$

This represents the present value of all future cash flows.

2.2 US Treasury Bonds

- Semi-annual coupon payments: $C_1 = C_2 = \ldots = C_{N-1} = C$
- Final payment: $C_N = C + \text{Face Value}$
- Example: 5% annual coupon bond $\Rightarrow C = \$2.50$ semi-annually for \$100 face value
- Considered "risk-free" (no default risk)

2.3 Corporate Bonds

Corporate bonds have the same structure as Treasuries but include default risk. The pricing formula becomes:

$$\text{Corporate Bond Price} = \sum_{i=1}^{N} C_i e^{-r(t_i)\cdot t_i} \cdot e^{-\lambda t_i}$$

where:

- $r(t_i)$ = risk-free interest rate
- λ = hazard rate (default intensity)
- $e^{-\lambda t_i}$ = probability of no default by time t_i

This can be simplified as:

$$\text{Corporate Bond Price} = \sum_{i=1}^{N} C_i e^{-R(t_i)\cdot t_i}$$

where $R(t_i) = r(t_i) + \lambda$ is the corporate interest rate curve.

3 Credit Rating System

Rating agencies (S&P, Moody's, Fitch) classify bonds by default risk:

Rating	Description
AAA to AA	Highest Quality
A+ to BBB-	Investment Grade
BB+ to CCC	Junk Bonds (High Yield)
Below CCC	Distressed

4 Mortgage-Backed Securities

4.1 Mortgage Structure

Traditional mortgages are self-amortizing with:

- Fixed monthly payments over 30 years
- Each payment = Interest + Principal reduction
- Early payments are mostly interest; later payments are mostly principal
- No balloon payment (unlike corporate bonds)

4.2 Securitization Process

1. Banks originate mortgages to homeowners
2. Mortgages are pooled and sold to Special Purpose Investment Vehicles (SPIVs)
3. SPIVs create mortgage-backed securities (MBS):
   - RMBS = Residential Mortgage-Backed Securities
   - CMBS = Commercial Mortgage-Backed Securities
4. Geographic diversification was supposed to reduce risk

5 Collateralized Debt Obligations (CDOs)

5.1 CDO Structure

CDOs pool multiple bonds (corporate or mortgage-backed) and create tranches:

Tranche	Payment Priority	Risk Level
Super Senior	1st	Lowest
Senior (AAA)	2nd	Low
Mezzanine (BBB)	3rd	Medium
Equity	Last	Highest

5.2 Waterfall Structure

- All cash flows from underlying bonds are pooled
- Payments flow from top to bottom (waterfall)
- Senior tranches get paid first
- Equity tranche absorbs first losses but gets highest returns

5.3 CDO-Squared (CDO^2)

To deal with "toxic waste" (risky equity tranches):

1. Create new SPIV with 100 equity tranches from different CDOs
2. Re-tranche this pool to create new "AAA" securities
3. This amplified systemic risk by creating correlation

6 Credit Default Swaps (CDS)

6.1 Basic Structure

- CDS Buyer: Pays periodic premiums, receives protection against default
- CDS Seller: Receives premiums, pays losses if default occurs
- Common maturities: 3, 5, 7 years
- Traded over-the-counter (OTC)

6.2 Synthetic Exposure

- CDS buyer is effectively "short" the underlying bond
- CDS seller is effectively "long" the underlying bond
- No need to own the underlying bond to buy/sell protection
- Created massive synthetic exposure to mortgage risk

7 The Housing Bubble (2003-2007)

7.1 Contributing Factors

- Low interest rates and easy credit
- "NINJA" loans (No Income, No Job, No Assets verification)
- Teaser rates: 0% initial rates jumping to 8-9% after 2 years
- Interest-only payments (principal actually increased)
- Massive demand drove house prices up 2-3x

7.2 The Crisis Unfolds (2007-2009)

1. Teaser rates reset to higher levels
2. Monthly payments became unaffordable
3. Mass defaults and foreclosures

4. House prices collapsed
5. Geographic diversification failed - subprime was everywhere

8 Early Warning Signs (2004-2006)

8.1 Market Observations

- Increase in subprime borrowers across all RMBS pools
- CDS protection on RMBS remained cheap (15-25 basis points)
- Smart money started buying CDS protection on subprime tranches
- Hedging strategy: Buy CDS on equity/mezzanine, sell CDS on senior tranches

9 The Gaussian Copula Model

9.1 Background and Development

The Gaussian Copula model became the industry standard for pricing CDOs in the early 2000s. Developed by David Li in 2000, it provided a tractable way to model joint default probabilities across multiple assets. The model's elegance and computational efficiency made it widely adopted, but its limitations would prove catastrophic.

9.2 Mathematical Foundation

9.2.1 Copula Theory Basics

A copula separates the dependence structure from the marginal distributions. For n random variables $U_1, U_2, \ldots, U_n$ with marginal distributions $F_1, F_2, \ldots, F_n$, the copula C satisfies:

$$F(x_1, x_2, \ldots, x_n) = C(F_1(x_1), F_2(x_2), \ldots, F_n(x_n))$$

9.2.2 Gaussian Copula Specification

For the Gaussian copula with correlation matrix Σ:

$$C_\Sigma(u_1, \ldots, u_n) = \Phi_\Sigma(\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_n))$$

where:

- Φ_Σ = multivariate normal CDF with correlation matrix Σ
- Φ^{-1} = inverse standard normal CDF
- u_i = marginal default probabilities

9.3 CDO Application

9.3.1 Asset Value Model

For each obligor i, model the standardized asset value:

- Asset value: $V_i \sim$ some distribution
- Standardized: $X_i = \frac{V_i - \mu_i}{\sigma_i}$
- Default threshold: $X_i < \Phi^{-1}(p_i)$ where p_i = default probability
- Joint distribution: $(X_1, \ldots, X_n) \sim$ Multivariate Normal with correlation Σ

9.3.2 Single Factor Model

To avoid the complexity of full $n \times n$ correlation matrix, practitioners used:

$$X_i = \sqrt{\rho_i} \cdot Z + \sqrt{1 - \rho_i} \cdot \epsilon_i$$

where:

- $Z \sim N(0, 1)$ = systematic risk factor (market factor)
- $\epsilon_i \sim N(0, 1)$ = idiosyncratic risk factor
- ρ_i = asset correlation with systematic factor
- All ϵ_i are independent, and independent of Z

  This implies pairwise correlation: $\text{Corr}(X_i, X_j) = \sqrt{\rho_i \rho_j}$
  For homogeneous portfolios (same ρ for all assets):

$$\Sigma = \begin{bmatrix} 1 & \rho & \rho & \cdots & \rho \\ \rho & 1 & \rho & \cdots & \rho \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \rho & \rho & \rho & \cdots & 1 \end{bmatrix}$$

9.4 Conditional Default Probability

Given the systematic factor $Z = z$, defaults become conditionally independent:

$$P(\text{Default}_i|Z = z) = \Phi\left(\frac{\Phi^{-1}(p_i) - \sqrt{\rho_i} \cdot z}{\sqrt{1 - \rho_i}}\right)$$

This allows efficient Monte Carlo simulation:

1. Draw systematic factor: $Z \sim N(0, 1)$
2. For each obligor, calculate conditional default probability
3. Simulate defaults independently given Z
4. Calculate portfolio losses and tranche payments

9.5 Base Correlation and Market Calibration

9.5.1 The Correlation Smile Problem

When calibrating to market prices, different tranches implied different correlation parameters - creating a "correlation smile" similar to volatility smiles in options. This indicated model misspecification.

9.5.2 Base Correlation

To address this, practitioners developed "base correlation":

- Instead of tranche correlations, use correlations for cumulative losses from 0% to each attachment point
- Base correlation for [0%, 3%] tranche, [0%, 7%] tranche, etc.
- Interpolate to price arbitrary tranches

9.6 The Correlation Problem

Key Insight: Higher correlation affects tranches differently.

For a simple 2-bond CDO with default probability p:

Tranche	Correlation = 0	Correlation = 0.5	Correlation = 1
Senior (2 defaults)	p^2	intermediate	p
Mezzanine (1 default)	$2p(1-p)$	intermediate	p
Equity (0+ defaults)	$1-(1-p)^2$	intermediate	p

Economic Intuition:

- **Low correlation:** Defaults are diversified away - senior tranches very safe
- **High correlation:** Either everyone defaults or no one does - senior tranches riskier
- **Mezzanine paradox:** Actually becomes *safer* with higher correlation

9.7 Model Limitations and Crisis

9.7.1 Static Correlation Assumption

The model assumed correlation remained constant over time and across market conditions. In reality:

- **Normal times:** $\rho \approx 0.1 - 0.3$ (geographical diversification works)
- **Crisis times:** $\rho \approx 0.7 - 0.9$ (national housing collapse)
- Model couldn't capture this regime switching

9.7.2 Tail Dependence

Gaussian copula has zero tail dependence:

$$\lim_{u \to 1} P(U_2 > u | U_1 > u) = 0$$

This means extreme events are modeled as independent, severely underestimating systemic risk.

9.7.3 The "Formula That Killed Wall Street"

The Gaussian copula's mathematical elegance masked its dangerous assumptions:

1. **Normal distributions:** Asset returns are not normally distributed
2. **Linear correlation:** Doesn't capture nonlinear dependencies
3. **Static parameters:** Correlation changes with market regimes
4. **Model uncertainty:** Overconfidence in a single model

9.8 Alternative Models

Post-crisis developments included:

- **t-Copula:** Allows for tail dependence
- **Archimedean Copulas:** Clayton, Gumbel copulas with different tail behavior
- **Regime-switching models:** Allow correlation to change over time
- **Factor models:** Multiple systematic risk factors

Critical Lesson: No single model should be trusted completely. Model risk must be explicitly managed through stress testing and alternative model validation.

10 Lessons Learned

1. **Model Risk:** Gaussian Copula underestimated tail risk and correlation breakdown
2. **Systemic Risk:** CDO-squared created dangerous interconnectedness
3. **Regulatory Gaps:** CDS market was unregulated and opaque
4. **Incentive Problems:** "Originate to distribute" model reduced due diligence
5. **Rating Inflation:** Agencies had conflicts of interest in rating structured products

11 Mathematical Appendix

11.1 Monte Carlo Simulation for CDO Pricing

1. Generate $Z \sim N(0,1)$ (common factor)
2. For each obligor i: Generate $\epsilon_i \sim N(0,1)$
3. Compute $X_i = \sqrt{\rho} \cdot Z + \sqrt{1-\rho} \cdot \epsilon_i$
4. Check defaults: $D_i = 1$ if $X_i < \Phi^{-1}(p_i)$, else $D_i = 0$
5. Calculate losses and tranche payments
6. Repeat simulation and average results

11.2 Recovery Rate Adjustment

In practice, bonds don't lose 100% of value upon default:

$$\text{Loss Given Default} = \text{Notional} \times (1 - \text{Recovery Rate})$$

Typical recovery rates: 30-40% for corporate bonds, 50-70% for mortgages.