IEORE4407: Game Theory Models of Operation

Lecture Notes
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Static Games with Complete Information

10 September 2025

- n players.
- Player i has action set A_i.
- Payoff: $u_i : A_1 \times A_2 \times \ldots \times A_n \to \mathbb{R}$
- $u_i(a_1, a_2, \ldots, a_i, \ldots, a_n)$ is the payoff to player i for choosing a_i.
- Actions are common knowledge.
- One-shot game.
- Each player picks an action "simultaneously" (without knowing others' choices).

	L	R
U	$9,9$	$0,10$
D	$10,0$	$4,1$

- Row player can choose U or D.
- Column player can choose L or R.

Dominant Strategies

The "best" choice never depends on what the other person does.

- Row player should pick D anyways.
- Column player should pick R anyways.
- We assume players are rational.
- Expected outcome: $(9,4)$
- This is the "Prisoner's Dilemma."

- If they interact: Row $\to U$, Column $\to L$.

For strategies:

$(s_1, s_2, \ldots, s_i, \ldots, s_n)$, s_i is the strategy for player i.

s_i^* is a dominant strategy for player i if:

$$u_i(s_1, s_2, \ldots, s_i^*, \ldots, s_n) > u_i(s_1, s_2, \ldots, s_i, \ldots, s_n)$$

for all $s_1 \in A_1, s_2 \in A_2, \ldots$.

Dominant Strategy Equilibrium

If a game is such that each player i has a dominant strategy s_i^*, then the game has a dominant strategy equilibrium $(s_1^*, s_2^*, \ldots, s_n^*)$.

Dominance Solvable

$\tilde{s}$ is a "dominated" strategy for player i if there is another strategy $\tilde{s_i}$ such that:

$$u_i(s_1, s_2, \ldots, \tilde{s_i}, \ldots, s_n) < u_i(s_1, s_2, \ldots, s_i, \ldots, s_n)$$

Static games

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Read: Game theory in supply chain analysis (survey paper). (Cachon & Netessine)

Definitions

Game

- A game is defined with n players ($n \geq 2$).
- Set of actions available to i: $A_1, A_2, \ldots, A_n$.
- Strategy space of i: $S_1, S_2, \ldots, S_n$.
- Payoff function: $u_1, u_2, \ldots, u_n$.

$$u_i : S_1 \times S_2 \times \cdots \times S_n \to \mathbb{R}$$

$$u_i(s_1, s_2, \ldots, s_i, \ldots, s_n) = \text{Payoff to } i \text{ when players choose } s_1, s_2, \ldots$$

- Depends on $(s_1, s_2, \ldots, s_n)$, not just on s_i.
- Could be different for different players.

Dominant Strategy

s_i is a dominant strategy if:

$$u_i(t_i, \ldots, \underline{s_i}, \ldots, t_n) > u_i(t_i, \ldots, \underline{s_i'}, \ldots, t_n)$$

$$\forall t_1 \in S_1, t_2 \in S_2, \ldots, t_n \in S_n \quad \text{where} \quad s_i \neq s_i'$$

Strictly Dominated Strategy

Let $t_i \in S_i, s_i \in S_i$.

t_i is strictly dominated by s_i if:

$$u_i(s_i, s_{-i}) \geq u_i(t_i, s_{-i}) \quad \forall s_{-i} \in S_{-i}$$

- A strictly dominated strategy may not exist.
- A rational player will never play a strictly dominated strategy.

	C	M	H
L	6/6	2/8	0/5
M	8/2	4/1	1/3
H	4/10	3/1	2/2

For row players, L is dominated.
For column players, C is dominated.

$\therefore$ Game is effectively

	M	H
M	4/1	1/3
H	3/1	2/1

Rationalizability and Beliefs

Belief of player $i \equiv$ some random $B_{-i} \subseteq S_{-i}$.

S_i : Player's best response to B_{-i}.

$$(u_i(s_i, B_{-i}) \geq u_i(s_i', B_{-i}) \quad \forall s_i' \in S_i)$$

$(s_1, s_2, \ldots, s_n)$ is rationalizable if s_i is a best response to some B_{-i} for each player i.

- Belief of i about j can be different from belief of k about j.
- Rationalizability allows for that.

Examples

Example 1:

	D	C
D	$-1/-1$	$9/0$
C	$0/9$	$-1/-2$

- C is dominant strategy for each player.
- D is dominated.
- (C, C) can be rationalized as N.E.

Example 2:

	O	F
O	$2/1$	$0/0$
F	$0/0$	$1/2$

- No dominant/dominated strategy.
- N.E. $= (0, 0)$ (pure strategies)
- Rationalizable strategies: all rationalizable (O, F), (F, O), etc.

Example 3:

	L	R
U	$8/8$	$5/9$
D	$9/5$	$6/6$

- D is dominant.
- U is dominated.
- R is dominant.
- L is dominated.

Example 4:

	L	M
V	$1/0$	$1/2$
H	$0/3$	$2/0$

- R is dominated.
- Only N.E. rationalizable.
- V is dominating, M is dominating.

More Games

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1 Cournot Competition

17 September 2025

1.1 Setup

- 2 firms: i and j (quantity competition)
- Marginal production cost $= c$ (cost to produce 1 item)
- Firms choose quantities q_i and q_j simultaneously
- Unit price $= a - (q_i + q_j)$

Question: What are equilibrium quantities chosen by the firms?

(In game theory context)

- Number of players = 2 (i and j)
- Strategy space:

$$\text{Firm } i : [0, \infty) \tag{1}$$
$$\text{Firm } j : [0, \infty) \tag{2}$$

Payoff for firm i: (Revenue - cost = Profit)

$$U_i(q_i, q_j) = q_i \cdot (\text{unit price} - \text{marginal cost}) \tag{3}$$
$$= q_i(a - q_i - q_j - c) \tag{4}$$
$$\therefore U_i(q_i, q_j) = q_i(a - q_i - q_j - c) \tag{5}$$

1.2 Finding Nash Equilibrium

(q_i^*, q_j^*) are Nash Equilibrium if:

$$U_i(q_i^*, q_j^*) \geq U_i(q_i, q_j^*) \quad \forall q_i^* \neq q_i \tag{6}$$
$$U_i(q_i^*, q_j^*) \geq U_i(q_i^*, q_j) \quad \forall q_j^* = q_j \tag{7}$$

Suppose firm j produces $\hat{q}_j$. Then q_i^* is the optimal reaction of firm i:

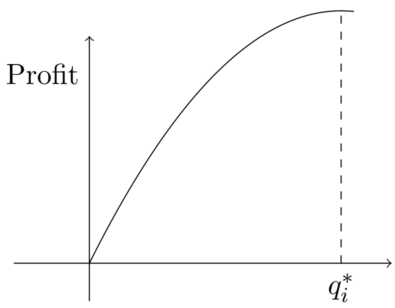
$$\max_{q_i} \left(q_i(a - q_i - \hat{q}_j - c) \right) \tag{8}$$

where $\hat{q}_j$ is a known fixed number (essentially a constant).

$$\pi(q_i) = q_i(a - q_i - \hat{q}_j - c) \tag{9}$$
$$\pi'(q_i) = (a - q_i - \hat{q}_j - c) + q_i(-1) = 0 \tag{10}$$
$$a - q_i - \hat{q}_j - c - q_i = 0 \tag{11}$$
$$\therefore q_i^* = \frac{a - \hat{q}_j - c}{2} \tag{12}$$

Since $\pi''(q_i) = -2 < 0 \Rightarrow q_i^*$ is a max.

If you plot a profit function:



Similarly, $q_j^* = \frac{a-q_i^*-c}{2}$

For both firms to get equilibrium:

$$q_i^* = \frac{a - q_j^* - c}{2} \tag{13}$$
$$q_j^* = \frac{a - q_i^* - c}{2} \tag{14}$$

Solving:

$$\therefore q_i^* = q_j^* = \frac{a-c}{3} \tag{15}$$

If both firms were owned by the same entity:

$$\max_{q_i, q_j} \{q_i(a - q_i - q_j - c) + q_j(a - q_i - q_j - c)\} \tag{16}$$

$$= \max_{q_i, q_j} \{(q_i + q_j)(a - c - (q_i + q_j))\} \tag{17}$$

Consider $q_i + q_j = q$:

$$\max_q \{q(a - c - q)\} \tag{18}$$

In that case:

$$q_i^{opt} + q_j^{opt} = \frac{a-c}{2} \tag{19}$$
$$\therefore q_i^{opt} = q_j^{opt} = \frac{1}{4}(a - c) \tag{20}$$

(Like prisoner's dilemma)

Even though $\frac{1}{4}(a-c)$ is better, if q_i plays that, j will play $q_j^* = \left(\frac{a-c}{2}\right) - \frac{1}{2}\left(\frac{a-c}{4}\right)$

$\therefore$ they will both end up playing sub-optimally, i.e., $\frac{(a-c)}{3}$

1.3 Why will q_j produce more?

q_j produces more, even though price will go down; the volume still gives q_j more profit than q_i.

So when we say:

$$q_i^* = \frac{a - q_j^* - c}{2} \tag{21}$$

So whatever i thinks j will do:

$$q_i^* = \left(\frac{a-c}{2}\right) - \frac{1}{2}(q_j^*) \tag{22}$$

Since q_j cannot be negative ($q_j \geq 0$):

$$\Rightarrow q_i^* \leq \left(\frac{a-c}{2}\right) \tag{23}$$

$\therefore$ i should never produce more than $\left(\frac{a-c}{2}\right)$

Similarly, $q_j^* \leq \left(\frac{a-c}{2}\right)$

We know $q_i^{opt} = q_j^{opt} = \frac{1}{4}(a-c)$

$\therefore$ min quantity i & j can produce: $\frac{1}{4}(a-c)$

(See full working in Tirole's textbook)

2 Bertrand Duopoly

("Price competition")

2.1 Setup

- 2 firms: i and j
- Prices: P_i and P_j where $a > 0$, $b > 0$, and c is the same

Demand:

$$\text{Firm } i = a - P_i + bP_j \tag{24}$$
$$\text{Firm } j = a - P_j + bP_i \tag{25}$$

Market Size: "a"

Production cost $= c$

Equilibrium prices: (P_i^*, P_j^*)?

Payoff for firm i: (price - cost)

$$\pi_i(P_i) = (a - P_i + bP_j)(P_i - c) \tag{26}$$

what i thinks choice j makes

$$\therefore P_i^* = \arg\max_{P_i \geq c} [(a - P_i + bP_j)(P_i - c)] \tag{27}$$

$$P_i^* = \frac{1}{2}(a + bP_j^* + c) \tag{28}$$
$$P_j^* = \frac{1}{2}(a + bP_i^* + c) \tag{29}$$

2.2 Solving the 2 equations

$$P_i^* = P_j^* = \frac{a+c}{2-b} \quad (b<2) \tag{30}$$

This is the equilibrium strategy.

3 Commons Problem

Public good: K agents $(k_1, k_2, \ldots, k_n)$ units

Agent i can claim any amount k_i

Utility for $i = \ln(k_i) + \ln\left(k - \frac{\sum k_j}{3}\right)$

log is concave $\Rightarrow$ these are good models of utility function (e.g., utility of money)

Let's try for 3 agents: (k_1^*, k_2^*, k_3^*)

Suppose agent j uses k_2^*, agent 3 uses k_3^*

Then agent 1's optimal solution:

$$\max_{k_1} \left[\ln(k_1) + \ln\left(k - k_1 - k_2^* - k_3^*\right)\right] \tag{31}$$

Taking derivatives:

$$\frac{1}{k_1} - \frac{1}{k - k_1 - k_2^* - k_3^*} = 0 \tag{32}$$

$$\therefore k - k_1 - k_2^* - k_3^* = k_1 \tag{33}$$

$$\Rightarrow \therefore k_1 = \frac{k - k_2^* - k_3^*}{2} \tag{34}$$

Assume all k_i are identical:

$$\therefore k^* = \frac{k - 2k^*}{2} \tag{35}$$

$$\Rightarrow \therefore k^* = \frac{k}{4} \tag{36}$$

In general: $k_i^* = \frac{k}{(n+1)}$ if we maximize utility for each agent

Optimal consumption for 3 agents (as a whole) :

Let $k_1 + k_2 + k_3 = K$
$\max_{k_1,k_2,k_3} \left[\ln(k_1) + \ln(k-K) + \ln(k_2) + \ln(k-K) + \ln(k_3) + \ln(k-K)\right]$

$\therefore k_i^* = \frac{k}{2n}$ maximizes overall utility for everybody

3.1 Notes

- Each agent's optimal problem means they give less weighting to overall consumption, because they only care about their own utility.

- If society wants to optimize overall welfare, there needs to be more weight on overall consumption and less on purely private benefits.
- agents consume more than they should compared to the optimal social solution. This is because of externalities — if part of the consumption cost is incurred by society as a whole, an individual tends to ignore it. As a result, we over-utilize the commons, since everyone is acting in their own self-interest without accounting for the shared damage.
- To regulate that, we can introduce tolls, penalties, or corrective taxes (Pigouvian taxes). These policies work by making private agents "internalize" the externality, i.e. aligning private incentives with the social optimum.

4 Final Offer Arbitration

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4.1 Setup

- Union $\sim$ arbitrator
- Firms
- Firm arguing over wages; wise arbitrator

Game:

- Firm proposes a wage offer w_F simultaneously
- Union chooses a wage after w_U

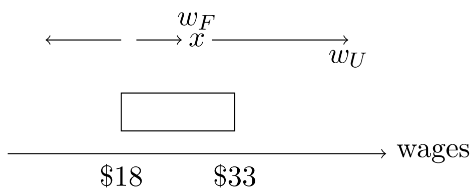
Arbitrator has an ideal settlement:

- Required to choose w_F or w_U

Arbitrator commits to choosing after doesn't in x.

From the union, believe x is a randomly distributed with distribution $F(x)$.

Example: $F(x)$



Then w_F is chosen (x is closer to w_F)

4.2 Equilibrium Analysis

What do the firm & union do in equilibrium?

Analysis: Suppose union picks w_U, firm picks w_F

$$P(w_U \text{ is chosen}) = P\left(x > \frac{w_U + w_F}{2}\right) = 1 - F\left(\frac{w_U + w_F}{2}\right) \tag{37}$$

$$P(w_F \text{ is chosen}) = P\left(x < \frac{w_U + w_F}{2}\right) = F\left(\frac{w_U + w_F}{2}\right) \tag{38}$$

$\therefore$ Expected wage agreement is: $G(w_U, w_F)$

$$= w_F \cdot F\left(\frac{w_U + w_F}{2}\right) + w_U \cdot \left(1 - F\left(\frac{w_U + w_F}{2}\right)\right) \tag{39}$$

If (w_U^*, w_F^*) is NE (Nash equilibrium), then:

w_F^* should minimize $G(w_U^*, w_F)$ over all w_F

w_U^* should maximize $G(w_U, w_F^*)$ over all w_U

$\therefore w_F$ should solve:

$$\min_{w_F} \left\{ w_F \cdot F\left(\frac{w_F + w_U^*}{2}\right) + w_U \cdot \left(1 - F\left(\frac{w_F + w_U^*}{2}\right)\right) \right\} \tag{40}$$

Similarly:

$$\max_{w_U} \left\{ w_F^* \cdot F\left(\frac{w_F^* + w_U}{2}\right) + w_U \cdot \left(1 - F\left(\frac{w_F^* + w_U}{2}\right)\right) \right\} \tag{41}$$

First order conditions:

At $w_F = w_F^*$:

$$0 = \frac{1}{2} \cdot w_F \cdot F'\left(\frac{w_F + w_U^*}{2}\right) + F\left(\frac{w_F + w_U^*}{2}\right) - w_U \cdot \frac{1}{2} \cdot F'\left(\frac{w_F + w_U^*}{2}\right) \tag{42}$$

At $w_U = w_U^*$:

$$0 = w_F \cdot F'\left(\frac{w_F + w_U^*}{2}\right) \cdot \frac{1}{2} - 1 + \left(1 - F\left(\frac{w_F + w_U^*}{2}\right) - w_U \cdot \frac{f(w_F^* + w_U)}{2}\right) \tag{43}$$

Equating both:

$$\frac{1}{2}(w_U^* - w_F^*) \cdot F'\left(\frac{w_U^* + w_F^*}{2}\right) = F\left(\frac{w_F^* + w_U^*}{2}\right) \tag{44}$$

$$\Rightarrow F\left(\frac{w_F^* + w_U^*}{2}\right) = \frac{1}{2} \tag{45}$$

At equilibrium, the avg of their choices would be at the median of the observations.

$$P(w_F) = P(w_U) = \frac{1}{2} \tag{46}$$

$$\therefore w_U^* - w_F^* = \frac{1}{F'\left(\frac{w_F^*+w_U^*}{2}\right)} \tag{47}$$

(density fn.)

Example: $F(x)$ is normally distributed $\sim N(m, \sigma^2)$

$$\therefore \frac{w_U^* - w_F^*}{2} = \frac{1}{f(m)} = \sqrt{2\pi}\sigma \tag{48}$$

$$\frac{w_U^* + w_F^*}{2} = m \quad \text{(for normal distribution, mean = median = } m\text{)} \tag{49}$$

Thus:

$$w_F^* = m - \left(\sqrt{\frac{\pi}{2}}\right)\sigma \tag{50}$$

$$w_U^* = m + \left(\sqrt{\frac{\pi}{2}}\right)\sigma \tag{51}$$

$\Rightarrow$ "gap" increased with uncertainty.

$\Rightarrow$ If they don't know what the arbitrator picks, firm & union tend to choose close to end points

5 Bertrand Competition (Variation)

2 firms: i & j, unit production cost $= c$

Prices: P_i, P_j

$$\pi(P_i, P_j) = \begin{cases} (a - P_i) \cdot (P_i - c) & P_i < P_j \\ 0 & (a - P_i) \cdot (P_i - c) \\ P_i > P_j & \\ \frac{(a-P_i)}{2} \cdot (P_i - c) & P_i = P_j \end{cases} \tag{52}$$

Note: $P_i > c$ can never be supported as a NE

$\Rightarrow P_i^* = P_j^* = c$ is the unique NE

5.1 Discrete Model

Suppose Ps are required to be in the set $\{c, c+\varepsilon, c+2\varepsilon, c+3\varepsilon, \ldots\}$

"ε" = increment

What are the Nash equilibria in this model?

(c, c) is a NE

$(c+\varepsilon, c+\varepsilon)$ is also a NE

(Though one firm can undercut to c, they reduce their profit to 0.)

6 Mixed Strategies (Nash Equilibria)

Exercise: Find a mixed strategy equilibrium

	U	D
Alice	0,0	0,-1
Bob	0,-10	-90,-6

Suppose Alice plays U with prob. (p), D with prob. $(1-p)$

Bob plays L with prob. q, R with prob. $(1-q)$

What is Bob's best response?

If Bob picks L, expected payoff:

$$p(0) + (1-p)(-10) = 10p - 10 \tag{53}$$

Bob picks R, expected payoff:

$$p(-1) + (1-p)(-6) = 5p - 6 \tag{54}$$

$\Rightarrow$ Bob's best response

$$10p - 10 > 5p - 6 \tag{55}$$

$$5p > 4 \tag{56}$$

$$p > \frac{4}{5} \tag{57}$$

$$\text{Best response} = \begin{cases} L & \text{if } p > 4/5 \\ R & \text{if } p < 4/5 \\ \{L, R\} & \text{if } p = 4/5 \end{cases}$$

Suppose Bob plays L with prob. q, R with prob. $(1-q)$

Alice's best response:

If Alice plays U expected payoff:

$$q(0) + (1-q)(0) = 0 \tag{58}$$

If Alice plays D expected payoff:

$$q(10) + (1-q)(-90) = 100q + 90 \tag{59}$$

Alice's best response:

$$\begin{cases} U & \text{if } q < 9/10 \\ D & \text{if } q > 9/10 \\ \{U, D\} & \text{if } q = 9/10 \end{cases} \tag{60}$$

	O	F
O	2,1	0,0
F	0,0	1,2

$(0, 0)$ (F, F) are pure strategy NE

Suppose Alice plays O with prob. (p), F with prob. $(1 - p)$

Bob's best response:

If bob plays $O = p(1) + (1 - p)(0) = p$

$$F = p(0) + (1 - p)2 = 2 - 2p \tag{61}$$

$$p > 2/3 \tag{62}$$
$$p < 2/3 \tag{63}$$

$$\text{Bob's best response} = \begin{cases} O & \text{if } p > 2/3 \\ F & \text{if } p < 2/3 \\ \{O, F\} & \text{if } p = 2/3 \end{cases}$$

Suppose Bob plays O with prob. q, F with prob. $(1 - q)$

If Alice picks $O = q(2) + (1 - q)(0) = 2q$

$$F = q(0) + (1 - q)1 = 1 - q \tag{64}$$

$$2q > 1 - q \tag{65}$$
$$q > \frac{1}{3} \tag{66}$$

$$\text{Alice's best response} = \begin{cases} O & \text{if } q > 1/3 \\ F & \text{if } q < 1/3 \\ \{O, F\} & \text{if } q = 1/3 \end{cases}$$

$\therefore p = \frac{2}{3}, q = \frac{1}{3}$ is N.E $\leftarrow$ mixed strategy Nash equilibria

$p = 1, q = 1$ is N.E

$p = 0, q = 0$ is N.E

7 Two-Person Zero Sum Games

$$a_{ij} = \begin{cases} \text{Payoff to Alice (from Bob) if} \\ \text{Alice plays strategy } i \text{ \& Bob} = j \end{cases} \tag{67}$$

		Bob		
		$y_1, y_2 \dots$	$\dots$	y_n
	x_1			
Alice	x_2		a_{ij}	
	$\vdots$			
	x_m			

$(a_{ij} - a_{ij})$ zero sum

Alice's $x_i^* = (x_1^*, x_2^*, \dots, x_m^*)$ are NE if & only if:

Bob's $y_j^* = (y_1^*, y_2^*, \dots, y_n^*)$ are NE if & only if

$$U(x, y) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} y_j^* \tag{68}$$

$\therefore x^*$ & y^* are NE if & only if:

[box=]align U(x, y*) $\leq U(x^*, y^*) \leq U(x^*, y)$

7.1 Alternative Approach

Fix Bob: $(y_1^*, y_2^*, \dots, y_n^*)$

Alice's expected payoff for playing strategy i (Alice plays i^{th} row):

$$= a_i y_1^* + a_2 y_2^* + \dots + a_{in} y_n^* \tag{69}$$

Alice's best response = $\max_{1 \leq i \leq m} \left\{ \sum_{j=1}^{n} a_{ij} y_j^* \right\}$

7.2 Extended

Let $V^* = \max \left\{ \sum_{j=1}^{n} a_{ij} y_j^* \right\}$

If for some k, $\sum_{j=1}^{n} a_{kj} y_j^* < V^*$ then $x_k^* = 0$

So: $\sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} y_j^* \right) x_i^* = \sum_{i=1}^{m} V^* x_i^* = V^*$

Also, $V^* = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} y_j^* \right) x_i^* \leq \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} y_j^* \right) x_i^*$

$$= \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} x_i^* \right) y_j^* \tag{70}$$

Bob's problem: (Bob plays j^{th} column)

For any $(x_1, \dots, x_m)$ that Alice chooses

Bob's payoff = $\sum_{i=1}^{m} a_{ij} x_i^*$

$\therefore$ Bob's response $w = \min_{1 \leq j \leq n} \{\sum_{i=1}^{m} a_{ij} x_i^*\}$

(Remember - zero sum game so if Alice = 9, Bob = -9)

7.3 Completing the Proof

$$\therefore W^* = \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} x_i^* \right) y_j^* \tag{71}$$

$$\geq \sum_{j=1}^{m} \left(\sum_{i=1}^{m} a_{ij} x_i^* \right) y_j^* \tag{72}$$

$\therefore (x^*, y^*)$ is a N-E if $\Rightarrow$

[box=] align U$_1(x, y^*) \leq U_1(x^*, y^*)$
$\forall (x_1, x_2, \ldots, x_m)$
s.t. $\sum x_i = 1$
$x_i \geq 0$

Bob:

$$U_2(x^*, y) \leq U_2(x^*, y^*) \tag{73}$$

$$\forall (y_1, \ldots, y_n) \tag{74}$$

$$\text{s.t. } \sum y_j^* = 1 \tag{75}$$

$$y_j \geq 0 \tag{76}$$

$$-U_1(x^*, y) \leq -U_1(x^*, y^*) \tag{77}$$

$$\Rightarrow U_1(x^*, y) \geq U_1(x^*, y^*) \tag{78}$$

[box=] align $\therefore U_1(x, y^*) \leq U_1(x^*, y^*) \leq U_1(x^*, y)$

7.4 Bob's Perspective

$$U_1(x^*, y^*) \leq U_1(x^*, y) \tag{79}$$

$$(\Rightarrow) \quad \leq \min_{y \in S_2} \{U_1(x^*, y^*)\} \tag{80}$$

$$\leq \max_{x \in S_1} \left\{ \min_{y \in S_2} \{U_1(x, y)\} \right\} \tag{81}$$

7.5 Alice's Perspective

$$U_1(x^*, y^*) \geq U_1(x, y^*) \tag{82}$$

$$(\Rightarrow) \quad \geq \max_{x \in S_1} \{U_1(x, y^*)\} \tag{83}$$

$$\geq \min_{y \in S_2} \left\{ \max_{x \in S_1} \{U_1(x, y)\} \right\} \tag{84}$$

$\therefore$ N-E for 2 person zero sum games:

$U_1(x^*, y^*) = \max_{x \in S_1} \min_{y \in S_2} \{U_1(x, y)\}$

7.6 Example with Payoff Matrix

		Bob		
		q_1	q_2	q_3
	P_1	5	1	1
Alice	P_2	3	0	8
	P_3	4	4	0

Alice's payoff:

$$U : 5q_1 + q_2 + q_3 \tag{85}$$
$$M : 3q_1 + 0 + 3q_3 \tag{86}$$
$$D : 4q_1 + 4q_2 + 0 \tag{87}$$

Min W:

$$W \geq 5q_1 + q_2 + q_3 \tag{88}$$
$$W \geq 3q_1 + 3q_3 \tag{89}$$
$$W \geq 4q_1 + 4q_2 \tag{90}$$

$$q_1 + q_2 + q_3 = 1 \tag{91}$$
$$q_1, q_2, q_3 \geq 0 \tag{92}$$

Bob's payoff/loss:

$$L : 5p_1 + 3p_2 + 4p_3 \tag{93}$$
$$C : p_1 + 4p_3 \tag{94}$$
$$R : p_1 + 3p_2 \tag{95}$$

Bob wants to min:

$$\min_{q_1,q_2,q_3} \{\max\{5q_1 + q_2 + q_3, 3q_1 + 3q_3, 4q_1 + 4q_2\}\} \tag{96}$$
$$q_i \geq 0 \tag{97}$$
$$\sum q_i = 1 \tag{98}$$

This is equivalent to: W

Bob's payoff/loss (zero sum):

$$L : 5p_1 + 3p_2 + 4p_3 \tag{99}$$
$$C : p_1 + 4p_3 \tag{100}$$
$$R : p_1 + 3p_2 \tag{101}$$

Bob's optimal loss:

$$\max_{p_1,p_2,p_3} \{\min\{5p_1 + 3p_2 + 4p_3, p_1 + 4p_3, p_1 + 3p_2\}\} \tag{102}$$
$$\sum p_i = 1 \tag{103}$$
$$p_i \geq 0 \tag{104}$$

∴ LP problem in this case:

Max V

$$V \leq 5p_1 + 3p_2 + 4p_3 \tag{105}$$
$$V \leq p_1 + 4p_3 \tag{106}$$
$$V \leq p_1 + 3p_2 \tag{107}$$

8 Sequential Games / Extensive Form Games

6 October 2025

8.1 Games that unfold over time

8.1.1 Ingredients:

1. Set of players N
2. Payoff function for each players - payoff depends on outcomes & outcomes depend on actions of all players
3. Sequence in which players move
4. Available actions when it is a players turn to move
5. Knowledge of a player when it's their turn to move
6. Moves by Nature (= Prob. over exogenous events)

9 Sequential Games / Extensive Form Games

Sequential games are games that unfold over time.

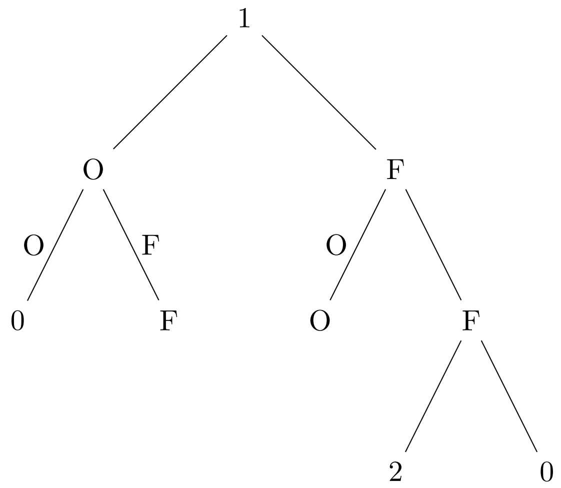
9.1 Key Ingredients

1. **Set of players** N
2. **Payoff** π_i for each player i
   - Payoff depends on outcomes; outcomes depend on actions of all players
3. **Sequence** in which players move
4. **Available actions** when it is a player's turn to move
5. **Knowledge of a player** when it's their turn to move
6. **Moves by nature** (probability over exogenous events)

9.2 Game Trees

Game trees are analogous to decision trees for single-player problems.

9.2.1 Example: Sequential Battle of the Sexes



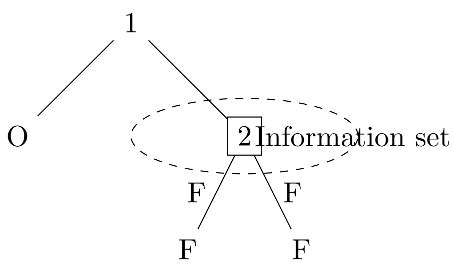
Note: In this example, player 1 moves first, choosing between O (Opera) and F (Football). Player 2 then observes player 1's choice and responds accordingly.

9.3 Information Sets

Information Set for player i: A partition of the nodes of player i such that player i cannot distinguish nodes within an information set.

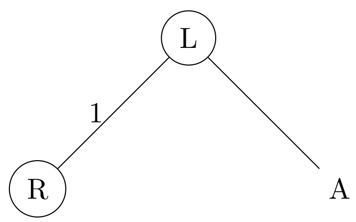
Note: One cannot distinguish nodes within an information set.

If nodes x and x' are in the same information set for player i, then their available actions are the same at these nodes.



10 Example Games

10.1 Game with Information Sets



Information sets for:

- Player 1: $\{A\}$
- Player 2: $\{B\}$ (assumed from context)

- Player 3: ϕ (Dummy player)
- Player 4: $\{D\}, \{E\}$

Terminal nodes: $\{C, F, G, H, I\}$

With terminal node payoffs shown in boxes at the leaves.

10.2 Deck of Cards Game

Game Description:

1. **Deck of cards**: Player 1 pulls out a card but does not look at it.
2. **After drawing**, player 1 chooses to either:
   - Call (C), or
   - Fold (F)
3. **Outcomes:**

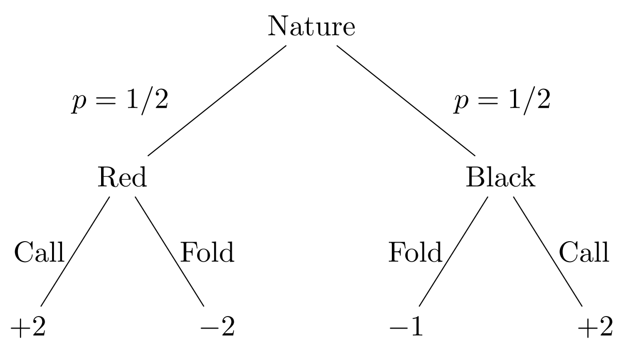
   [label=(c)]

   (a) If he folds, he pays $1 to player 2

   (b) If he calls, he pays $2 to player 2 if the card drawn is black

   (c) If he calls, he receives $2 from player 2 if the card drawn is red

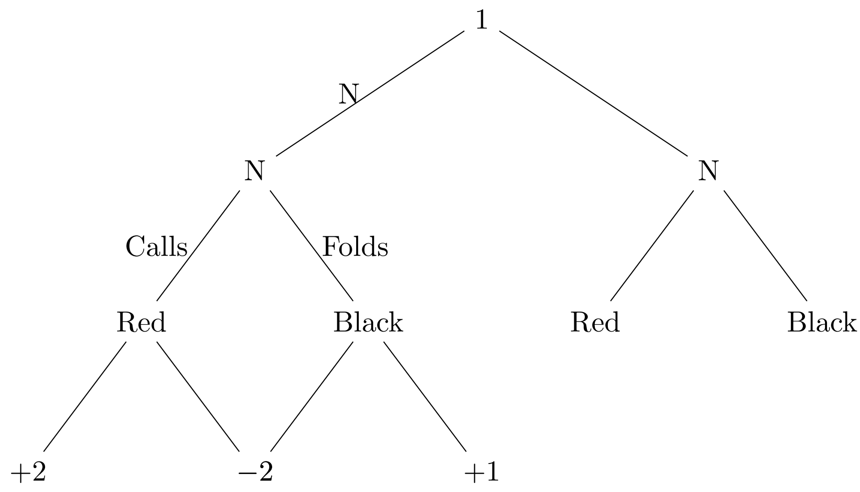
10.2.1 Game Tree



Note: Technically only 2 strategies for player 1 (they don't know if card is R/B; they can only C/F). Player 2 is a dummy player.

10.3 Same Game with Different Structure

Alternative representation where player 1 cannot observe nature's move:



11 Solution Concepts

11.1 Pure Strategy

A **pure strategy** is a complete plan of what player i would do, i.e., pick an action.

Formally: $S_i : H_i \to A_i$

As each $h_i \in H_i$, pick an action from $A_i(h_i)$.

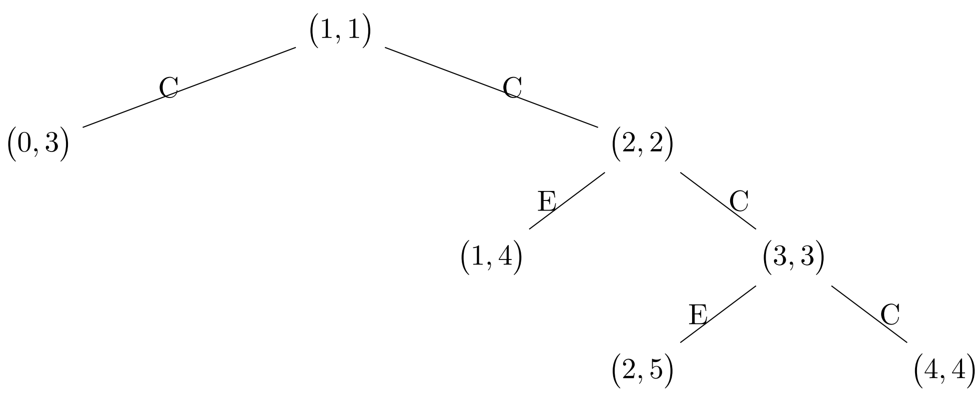
12 Extensive Form Games

13 October 2025

12.1 Properties

1. **1 has 6 descendants**: 2, 3, ...
2. **2 has 4 strategies**
3. Player assignments at different nodes

12.2 Centipede Game



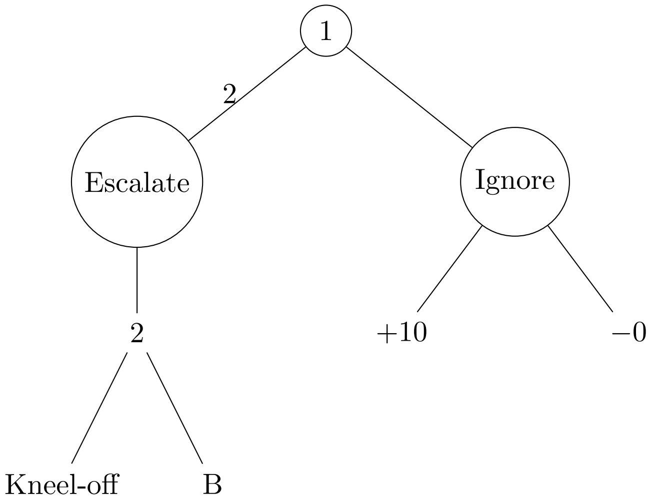
Where C = Continue, E = End

Note: $ keeps increasing.

Backward induction is very robust here because the game should theoretically end at the first step, but if it would remain at 2...

13 Example: Two Countries Game

Countries "1" and "2" game example:



Simultaneous move game:

$-5, \gamma -5$	$-100, d -100$
R $-5, -5$	$-100, -100$

D

Everybody loses.

13.1 Equivalent Normal Form Game

If they get to node 1, then $E \to 2N$, which is a Nash Equilibrium (NE).

	γ	d
R	$-5, -5$	$-100, -100$
D	$-100, -100$	$-100, -100$

(R, γ) is NE

(D, d) is NE

14 Subgame Perfect Nash Equilibrium

14.1 Definition

Subgame perfect NE: A Nash equilibrium where the strategy profile induces a Nash equilibrium in every subgame.

I want to have equilibria that are credible — not just the original game, but from any game from a subtree.

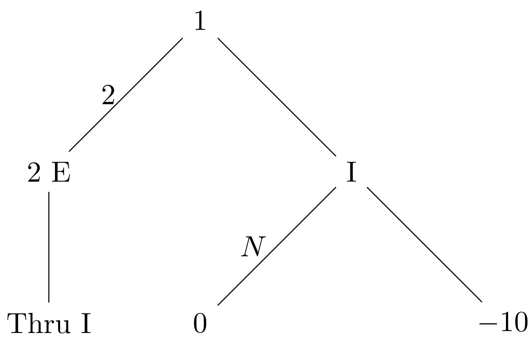
14.2 Example Analysis

	Nk	Nd	Bk	Bd
IR	0,0	0,0	0,0	0,0
ID	0,0	0,0	0,0	0,0
ER	$-5, -5$	$-100, -100$	$-10, -10$	$-10, -10$
ED	$-100, -100$	$-100, -100$	$-10, -10$	$-10, -10$

Where:

- I = Ignore
- E = Escalate
- R = Retaliate
- D = Don't retaliate
- N = Negotiation
- k = kneel
- B = Bomb

14.3 Additional Analysis



Notes:

- 2 picks N (choosing N based on payoffs)
- 1 picks I (ignoring based on backward induction)

15 Cournot Game

15.1 Setup

2 firms - each firm produces production cost c.

1. Firm 1 picks q_1
2. Firm 2 observes q_1 and picks q_2

Price in market: $a - q_1 - q_2$

Equilibrium strategies for both firms?

Recall: In simultaneous games, $q_1^* = q_2^* = \frac{a-c}{3}$

15.2 Analysis

Suppose firm 1 picks $q_1 \sim$ — what is the optimal choice of firm 2?

Firm 2's problem: $\max_{q_2} \{q_2(a - q_1 - q_2 - c)\}$

This is the "profit"

$$q_2^* = \frac{a - c - q_1}{2}$$

Strategy of firm 2: $q_2^* = f(q_1) = \frac{a-c-q_1}{2}$

Note: Strategy of firm 2 is a function that maps firm 1's action (a number, real) to an action.

15.3 Finding Firm 1's Optimal Choice

Firm 1 knows 2 is rational and will choose $\left(\frac{a-c-q_1}{2}\right)$.

$\therefore$ It's optimal choice is:

$$\max_{q_1} \left\{ q_1 \left(a - c - q_1 - \frac{a - c - q_1}{2} \right) \right\}$$

$= \max_{q_2} \left\{ a_1 \left(a - c - q_1 - \frac{a-c-q_1}{2} \right) \right\}$

$$\therefore q_1^* = \frac{a - c}{2}$$

$$\therefore q_2^* = \frac{a - c}{2} - \frac{a - c}{2} = \frac{a - c}{4}$$

$$q_{1,2}^* = \frac{a - c}{4}$$

15.4 Equilibrium Strategies

$$q_1^* = \frac{a - c}{2}$$

$$f_2(q_1) = \begin{cases} \frac{a-c-q_1}{2} & q_1 \leq a - c \\ 0 & q_1 > a - c \end{cases}$$

16 Example: Two-Stage Game

16.1 Stage 1 ($t = 1$)

	m	f
M	$4, 4$	$-1, 5$
F	$5, -1$	$1, 1$

16.2 Stage 2 ($t = 2$)

	ℓ	g
L	$0, 0$	$-4, -1$
G	$-1, 4$	$-3, -3$

16.3 Question

Can we support (M, m) as part of an equilibrium outcome?

16.4 "Carrot and Stick Approach"

If (M, m) is played in stage 1 ($t = 1$), pick (L, ℓ) otherwise pick (G, g).

(Penalize players from deviating)

Strategies: Player 1 plays M

If player 2 picks m and then ℓ, $\therefore$ payoff $= 4 + 0 = 4$

If player 2 picks f and then g, $\therefore$ player 1 plays G, $\therefore$ payoff $= 5 + (-3) = 2$

Note: This would not be an outcome if this was a single stage.

16.5 Transformed Game Matrix

16.5.1 At $t = 1$

	m	f
M	$4, 4$	$-1, 5$
F	$5, -1$	$1, 1$

16.5.2 At $t = 2$

	ℓ	g
L	$4, 4$	$-1, 5$
G	$5, -1$	$(1, 1)$ ✓ NE

So in essence the game becomes:

	m	f
M	$5, 5$	$0, 6$
F	$6, 0$	$(2, 2)$

F dominates M $\therefore$ $(2, 2)$ is the Nash equilibrium.

(Penalty for deviating) does not work because there is a unique equilibrium.

We don't possess a "tool" to penalize the other player with.

17 Summary

1. If you pick a NE of each stage game in isolation

   $\rightarrow$ can be supported as a subgame perfect NE

2. "Carrot & stick game" —

Non equilibrium outcomes can be supported as a subgame perfect equilibrium outcome in multistage games for a large enough discount factor.

18 Repeated Games

20 October 2025 $*$ stage game — repeated $t = 1, 2, \ldots, T$ (T can be ∞)

Stage game could be a normal form game.

Players utility: $u_1, u_2, \ldots$

$$\text{sum: } \sum v_t \quad \text{(could be infinite)}$$

Discounting: discount factor δ^t

$$\text{reward} = \frac{\infty}{\sum_{t=1}^{\infty} \delta^{t-1} v_t} \quad 0 \leq \delta \leq 1$$

Interpretation:

- Earlier rewards are worth more than later rewards
- $(1 - \delta^t)$ = probability of terminating at each stage

18.1 Equilibria in Infinitely Repeated Games

Pure strategy: A choice of action at each decision point

19 Example: Infinitely Repeated Prisoner's Dilemma

	m	f
M	$4, 4$	$-1, 5$
F	$5, -1$	$1, 1$

Possible strategies:

1. Play F every stage
2. Play F at odd t, M at even t
3. Play M initially, if the other player plays F then plays F otherwise M.
4. Play M initially, if the other player ever plays F, then pick F forever (otherwise M)

   ↑ **Grim / Trigger strategy**

   (once you lose trust it's lost forever)

20 "Conditional" Strategies

Use strategies in later stage games to support good behavior in earlier stage games.

20.1 Payoffs

Payoffs: $v_1, v_2, \dots, v_T, \dots$

Discounted payoff: $\sum_{t=1}^{\infty} \delta^{t-1} v_t = \cdot V$

"net present value of payoff"

The average payoff of $[v_1, v_2, \dots, v_T, \dots]$

$$= [\overline{v}, \overline{v}, \dots, \overline{v}, \dots]$$

Earning $\overline{v}$ in each step gives a total discounted payoff:

$$V = \left(\sum_{t=1}^{\infty} \delta^{t-1} \overline{v} \right) = \frac{\overline{v}}{1-\delta}$$

So, $\therefore \overline{v} = V \cdot (1-\delta)$

21 Repeated Prisoner's Dilemma Analysis

	m	f
M	$4,4$	$-1,5$
F	$5,-1$	$1,1$

Can (M, m) be sustained as an equilibrium in an infinitely repeated game?

21.1 "Carrot & Stick" Theory (Grim Trigger)

Row Player:

Stick: Play F following any outcome other than (M, m) at every earlier stage.

Carrot: If entire past history $= \{(M, m), (M, m) \dots\}$, then play M.

21.1.1 Column Player's Response

They play $f \therefore$ following grim trigger

$$5 + \frac{1 + 1 + \dots}{t = 0} = \frac{5 + \delta^1}{1-\delta}$$

They play M:

$$\therefore 4 + \frac{4\delta + 4\delta^2 + \dots}{1} = \frac{4}{1-\delta}$$

21.2 Condition for Cooperation

$$\frac{4}{1-\delta} > \frac{5 + \delta^1}{1-\delta}$$

$$\therefore 4 > 5 - 4\delta$$

$$4\delta > 1$$

$$\delta > \frac{-1}{4}$$

As long as $\delta \geq \frac{1}{4}$, the additional dollar you get by playing f does not equate to losing \$3 every stage thereafter.

$\therefore$ if $\delta > \frac{1}{4}$, then playing M is optimal.

Same for row player's side (by symmetry).

$\therefore$ this is subgame perfect NE as long as $\delta > \frac{1}{4}$.

22 Bargaining Games

22 October 2025

22.1 Folk Theorem

22.1.1 I. Ultimatum Game — 2 players split a dollar

Player 1 offers split $(s, 1-s)$

Player 2 accepts/rejects

$$S_1 = [(s, 1-s) \mid 0 \leq s \leq 1]$$

$$S_2 = \begin{cases} \text{Accept all offers above } x \\ \text{reject all offers below } x \end{cases}$$

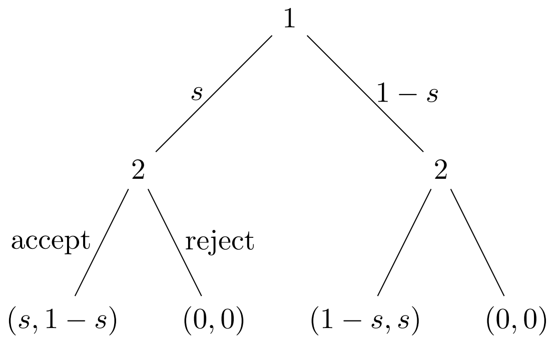
Possible strategies:

$\Rightarrow$ 1: offers $(x^*, 1-x^*)$

2: $\begin{cases} \text{accepts all offers } 1-x^* \text{ and above} \\ \text{rejects all below } 1-x^* \end{cases}$

Assumption: In case of tie, players accept rather than reject.

22.1.2 Game Tree Representation



Conditions:

- $s \geq 0.99$
- $1 - s \geq 0.01$

22.1.3 II. Bargaining Game (everything is common knowledge)

Stage 1: Player 1 offers a split $(s_1, 1 - s_1)$

Player 2 accepts | rejects

- s_1 — game proceeds to
- $1 - s_1$ — stage 2

Stage 2: Player 2 offers $(s_2, 1 - s_2)$

Player 1 accepts | rejects

- s_2 — game proceeds to
- $1 - s_2$ — stage 3

Stage 3: Player 1 offers split $(s_3, 1 - s_3)$

Player 2 accepts or rejects

$$\begin{cases} s_3 \\ 1 - s_3 \end{cases}$$

$$\begin{bmatrix} \hat{s} \\ 1 - \hat{s} \end{bmatrix}$$

$\hat{s}$ = some option (fixed), known to all from day 1

22.2 Optimal Offer Analysis

Optimal offer of player 1 on day 3 is $s_3 = \hat{s}$

ζ then player 2 should offer $s_2 \leq \hat{s}$, but player 1 would reject $s_2 < \hat{s}$.

ζ player 1 should offer $s_1 \geq \hat{s}$.

(This is all assuming value of money stays constant)

22.2.1 Impose Time Value of Money (discount factor δ)

Day 3: same choice $\hat{s}$

Day 2: player 2 should offer $s_2 = \delta^* \hat{s}$

$$1 - s_2 = (1 - \delta\hat{s})$$

$\therefore$ player 1 would accept any offer that gives them at least $\delta\hat{s}$.

Day 1: player 1 should offer

If s_1 is such that $1 - s_1 \leq \delta(1 - \delta\hat{s})$, then player 2 rejects.

Otherwise player 2 accepts. $\therefore$ optimal offer:

$$s_1 \leq 1 - \delta(1 - \delta\hat{s})$$

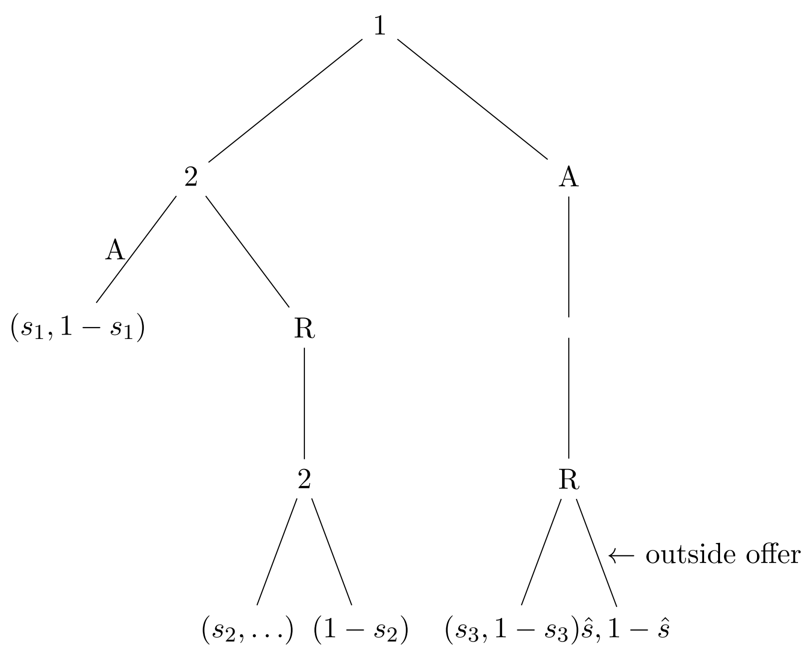
22.3 Extended Game Tree

$$\therefore s_1^* = 1 - \delta + \delta^2\hat{s}$$

If player 1 chooses $s_1 > s_1^*$, the player 2 rejects

and at day 2, player 1 gets $\delta^2\hat{s}$

and $\delta^2\hat{s} < 1 - \delta + \delta^2\hat{s}$



22.4 Summary of Bargaining Game Results

Player 1 gets: $1 - \delta + \delta^2\hat{s}$

Player 2 gets: $\delta - \delta^2\hat{s}$

$\therefore$ player 1 does better if δ close to 1 (e.g., $\delta = 0.95$)

$$\therefore 1 - \delta + \delta^2 \hat{s} \overset{?}{>} \delta - \delta^2 \hat{s}$$

$$\frac{1 - \delta + \delta^2}{2} > \frac{\delta - \delta^2}{2}$$

$$\hat{s} = \frac{1}{2}$$

$$\hat{s} = 1$$

$$1 - \delta + \delta^2 > \delta - \delta^2$$

23 Folk Theorem

Let G be a game of complete info. Let $(v_1^*, v_2^*, \ldots, v_n^*)$ be the payoffs from a NE of G.

Feasible payoff: Convex hull of payoffs.

	m	f
M	$(4,4)$	$(-1,5)$
F	$(5,-1)$	$(1,1)$

NE: $(1,1)$

[Diagram shows feasible payoff region $(4,4)$ to $(5,-1)$ to $(1,1)$ with shaded convex hull]

$(v_1, \ldots, v_n)$ can be achieved as the average payoff in a subgame perfect NE for δ large enough if $\Rightarrow$ $(v_1, \ldots, v_n)$ is feasible.

$$\ell : v_i \to v_i$$

24 Bayesian Games

27 October 2025 n players

Strategy spaces: $s_i, \ldots, S_n$

For player i:

i's payoff $= u_i(s_1, \ldots, s_n)$

Now — many types of player i:

$$u_i(s_1, \ldots, s_n; t) = \text{payoff for player } i \text{ of type } t, \text{ when the players use } s_1, \ldots, s_n \text{ resp.}$$

$$T_i = \text{type space for } i$$

$$t_i = \text{realized type of } i$$

$$S_i = \text{strategy space for } i$$

$$s_i = \text{strategy chosen by } i$$

$\Rightarrow T_1, \ldots, T_n$ — Nature draws a type for each player $(t_1, \ldots, t_n)$

i informs i of their type t_i:

Distribution of $(t_1, \ldots, t_n)$ is known

24.1 Beliefs

Based on their type, they update their beliefs:

$$\text{belief}_i : P_i(t_{-i} \mid t_i)$$

$\downarrow$ uncertainty that i has about types of other players.

$\rightarrow$ is game of different types of players. Based on my type I have some beliefs of other types.

$$P_i(t_{-i} \mid t_i) = \frac{P(t_{-i}, t_i)}{P(t_i)} = \frac{P(t_{-i}, t_i)}{\sum_{t'_i \in T_i} P(t_{-i}, t'_i)}$$

24.2 Bayes Nash Equilibrium

Bayes NE:

$(s_1^*, s_2^*, \ldots, s_n^*)$ is BNE if for $\forall i$, each $t_i \in T_i$:

$$\sum u_i(s_1^*(t_1), s_2^*(t_2), \ldots, s_i^*(t_i), \ldots, s_n^*(t_n)) \cdot P(t_{-i} \mid t_i)$$

$$t_i \in T_i$$

$$\sum u_i(s_1^*(t_1), \ldots, s_n^*(t_n), \ldots, s_i(t_i), \ldots) \cdot P(t_{-i} \mid t_i)$$

$$s_i(t_i), \ldots, s_n(t_n)$$

25 First-Price Auction

2 players, 1 object

Each player has value uniformly distributed in $[0, 1]$

$\therefore$ Type of a player can be = how much they value an object.

Valuations are drawn independently.

Game: First price sealed bid for object.

Highest bid wins object, receives their bid. Other bid pays nothing and gains nothing.

Payoff:

$$\begin{cases} v-b & \text{if they value obj at } v \\ 0 \end{cases}$$

Type space for bids, 1 & 2: $[0,1]$

Suppose my value is v and I bid b.

Expected payoff: $\max_{0\leq b\leq v}\{(v-b)\cdot P_i(b \text{ wins})\}$

25.1 Solution

Guess that player 2 bids $\frac{\sqrt{v}}{2}$ $(v<1)$, his value is $\hat{v}$

$\therefore$ my bid $b \geq \frac{\sqrt{v}}{2}$

$$\therefore \max_{0\leq b\leq v}\left\{(v-b)\cdot P\left[\hat{v}\leq\frac{b}{2}\right]\right\}$$

$$=\max_{0\leq b\leq v}\left\{(v-b)\cdot P\left[\frac{\hat{v}}{2}\leq\frac{b}{2}\right]\right\}$$

$$=\max_{0\leq b}\left\{(v-b)\left(\frac{b}{2}\right)\right\}$$

$$\Rightarrow b^*=\frac{v}{2}$$

26 Cournot Example: "Quantity Example"

(Simultaneous move game)

Firm i: q_i, c_i

Firm j: q_j $\begin{cases} C_{jH} & \text{prob. } p \\ C_{jL} & (1-p) \end{cases}$

(unit) selling price $= a - q_i - q_j$

(section 3.1 — Gibbons book)

27 Double Auction

1 buyer — wants to buy a house (indicated by $\uparrow$)

1 seller — wants to sell house (indicated by $\downarrow$)

iid $\cup[0,1]$ valuations.

$$v_b = 0.63 \qquad v_s = 0.49$$

So buyer/seller can come to an agreement that makes both happy, but not observable.

What is the efficient outcome?

$$\begin{cases} \text{sale} & \text{if } b > s \\ \text{not sale} & b < s \end{cases}$$

27.1 Revelation Principle

Particular equilibrium of some game.

There is another game in which type space = action space.

27.1.1 Double Auction

Truthful direct mechanism: (v_b, v_s)

- Buyer & seller submit sealed bids (b) (s)

$*$ If $(b - s) > \frac{1}{4}$, then trade occurs at

$$p = \frac{1}{3}(b + s) + \frac{1}{6}$$

(average of value + constant)

Otherwise no trade.

27.2 Environments Where Lack of Transparency Causes Problems

27.2.1 "Winner's Curse"

2 players, 1 obj

buyer seller

Object would be good, mediocre, bad (G) (M) (B)

Seller knows quality.

Buyer (B) believes quality to be $G \mid M \mid B$

Common knowledge:

$$v_s(G) = 30 \qquad v_B(G) = 34$$

$$v_s(M) = 20 \qquad v_B(M) = 24$$

$$v_s(B) = 10 \qquad v_B(B) = 14$$

You'd only really pay \$10 to start ← to know quality of object.

Buyer names a price   Seller accepts/rejects } market for lemons

28 Bayesian Auctions

29 October 2025

28.1 Private Value Auctions

28.1.1 (1) Second Price Auctions — Vickrey "truthful mechanism"

- Submit sealed bids
- Highest bid wins — but money charged is second highest bid

∴ it is a dominant strategy to bid your value.

Bidder 1: value x, cost c highest competing bid

Bid $b > x$: Payoffs

$$\begin{cases} x - c & \text{if } c < x \\ 0 & \end{cases}$$

Bid $b < x$: Payoffs:

$$\begin{cases} x - c & c < b \\ 0 & b < c < x \\ 0 & c > x \end{cases}$$

Note: Money left on the table when $b < c < x$.

Bid $b > x$: Payoffs =

$$\begin{cases} x - c & c < x \\ x - c & x < c < b \\ 0 & c > b \end{cases}$$

(so when $x < c < b$)

28.1.2 Which Format is Better for the Seller — First or Second Price Auctions?

Linearity of expectation:

$$R = x_1 + x_2$$

$$E[R] = E[x_1] + E[x_2]$$

2 bidders first price auction: $b_1 = x_1/2$, $b_2 = x_2/2$

density: $\int_0^1 \frac{x_1}{2}(x_1)(1-x_1)$ (uniform dist)

$$= \frac{x_1^3}{3}\Big|_0^1 = \frac{1}{6}$$

On average, seller collects $\frac{1}{6}$ from bids.

$$\therefore E(R_1) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$$

28.2 Second Price Auction

Values x_1, x_2

$$\text{Prob(bidder 1 wins with value } x) = x$$

$$E[\text{bidder 2} \mid \text{bidder 1 wins with value}(= b_2) = x] = \frac{x}{2}$$

What player has to pay:

$$E(\text{payment}) = E(\text{bid of player}_2)$$

What player 1 has to pay is bid$_2$, which is uniformly distributed between 0 & x:

$$\therefore E(\text{bid}_2) = \frac{x}{2}$$

$\therefore$ again $E[R] = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$

28.3 Expected Payments

28.3.1 First Price:

$$= \text{Prob}(x_1 > x_2) \cdot E[R^1(x_1)]$$

$$= x_1 \cdot \frac{1}{2} x_1$$

28.3.2 Second Price:

$$= \text{Prob}(x_1 > x_2) \cdot E[R(x_2 \mid x_1 > x_2)]$$

$$= x_1 \cdot \frac{1}{2} x_1$$

28.4 Generalizations

N bidders

Values $x_1, \ldots, x_n$ iid F on $[0, \omega]$

28.4.1 Expected Payment

$$m_i(x_i) = \text{Prob}(x_i > \max_{j \neq i} x_j)$$

$$E\left[\max_{j \neq i} x_j \mid x_i > \max_{j \neq i}\right]$$

Distribution of $y_i = \max_{j \neq i} x_j$